113 Class Problems: Polynomial Factorization II

1. Determine if the following polynomials in $\mathbb{Q}[x]$ are irreducible.
   (a) $14 + 42x - 90x^3 - 9x^6$
   (b) $1 + 3x^2 + 5x^3 - \frac{1}{3}x^4$
   (c) $1 + x + x^3$

   Solutions:

a) Yes. Eisenstein with $\pi = 2$

irreducible by Eisenstein $\pi = 3$

b) Yes $1 + 3x^2 + 5x^3 - \frac{1}{3}x^4 = \frac{1}{3}(3 + 9x^2 + 15x - x^4)$

$\Rightarrow 1 + 3x^2 + 5x^3 - \frac{1}{3}x^4$ irreducible.

c) Yes

$\deg(1+x+x^3) = 3 \Rightarrow$ $1+x+x^3$ irreducible in $\mathbb{Q}[x]$ $\Leftrightarrow$ $\nexists\, \alpha \in \mathbb{Q}$ such that $f(\alpha) = 0$

$1 + x + x^3$ monic $\Rightarrow$ $f(\alpha) = 0$ for $\alpha \in \mathbb{Q}$ $\Rightarrow \alpha \in \mathbb{Z}$

$f(x) = 1 + x + x^3 \Rightarrow f'(x) = 1 + 3x^2 > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x)$ monotonic increasing $\quad f(-1) = -1, f(0) = 1$

$\Rightarrow \nexists\, \alpha \in \mathbb{Z}$ such that $f(\alpha) = 0 \Rightarrow 1 + x + x^3$ irreducible

2. Let R be a UFD which is not a field. Prove that $Frac(R)$ is not algebraically closed.

   Solutions:

exists because R not a field

Let $\pi \in R$ be irreducible

Eisenstein's Criterion $\Rightarrow$ $\pi + \pi x + x^2 \in Frac(R)[x]$

is irreducible $\Rightarrow$ $Frac(R)$ not algebraically closed

3. (Hard) Let F be a field. Let H be a finite subgroup of the multiplicative group of units F^*. Prove that H is cyclic. Must this be true if F is not a field?

Solutions:

$H \subset F^*$ finite Abelian group. $|H| = n$

Assume H <u>not</u> cyclic

Structure Theorem $\Rightarrow$ $\exists$ $m < n$ such that $a^m = 1_F$ $\forall a \in H$

Example $n = 125 = 5^3 \Rightarrow (H, \times) \cong \begin{cases} \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z} & m = 5^2 \\ \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} & m = 5 \end{cases}$

$\Rightarrow$ Every $a \in H$ is a root of $x^m - 1_F$.

F a field $\Rightarrow$ number of distinct roots of $x^m - 1_F$ in F $\leq m < |H|$

<u>Contradiction</u> Hence H is cyclic